Division by Zero

The Introduction of Hyper-complex and Meta-complex Numbers

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Abstract

To allow for division by zero, the present study introduces the zero-unit z, which, in combination with real and complex numbers, leads to the sets of hyper-complex and meta-complex numbers. Some properties of these numbers are discussed, and an initial introduction to arithmetic with numbers is provided. The study is not complete, but is merely intended as a starting point for possible further research, since the introduction of new number sets might open up new perspectives.

1. Introduction

Complex numbers were introduced in mathematics as a way to solve mathematical problems. These problems arose when taking the square root of negative numbers, e.g., in solving quadratic equations. Such equations with negative numbers under square roots seem to have no meaningful solutions within the domain of real numbers. To overcome this obstacle, the concept of the imaginary number i was introduced. The imaginary unit i is defined as:

$$i = \sqrt{-1}$$

This means that:

$$i^2 = -1$$

This definition leads to the formation of complex numbers, represented in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit. The imaginary number i is an abstract concept. Concretely, i has no equivalent in real numbers because there is no real number whose square is equal to -1. Although i doesn't "exist" in the way real numbers do (it's not a real number), it has very concrete and valuable mathematic significance and applications. It serves as a tool to solve mathematical problems that would otherwise be unsolvable. Complex numbers found their way into various branches of mathematics and science. They are indispensable in quantum mechanics (e.g., in the Schrödinger equation).

The square root of negative (real) numbers is not the only undefined ("forbidden") operation in mathematics. Division by zero is also undefined.[1] But what prevents us similar to how the

[1] Logarithms of negative numbers are not defined in real numbers either, because they cannot be interpreted as exponents of positive numbers to produce a negative result. However, introducing a unit for logarithms of negative numbers is beyond the scope of this text. To allow for logarithms of negative numbers, the unit $l = log(-1)$ might be introduced. Numbers of the form al, where $a \in \mathbb{R}_0^+$ could be called l-numbers. For every $a \in \mathbb{R}_0^+$, the following holds: $log(-a) = log(-1.a) = log(-1) + log(a) = l + log(a)$. That is the sum of an l-number and a real number. The combination of l-numbers with real, imaginary, and zero numbers would lead to an explosion of dimensions and numbers of the form $a + bi + cz + diz + fl + gil + hzl + jizl$. These are eight-dimensional numbers similar to octonions (Graves and Hamilton).

number i was introduced to enable square rooting of negative numbers, from introducing a number that makes division by zero possible? What are the consequences and potential applications of such an introduction? That is the subject of the present study.

2. The zero-unit z

Let us define the **zero-unit** z as:

$$z = \frac{1}{0}$$

This number does not belong to the set of real numbers $\mathbb{R}$. Division by zero in $\mathbb{R}$ results in an indeterminacy. The $\lim_{x\to 0}\frac{1}{x}$ is not defined in $\mathbb{R}$; for positive real values approaching 0, $\frac{1}{x}$ gets closer to ∞, while for negative values approaching 0, $\frac{1}{x}$ gets closer to $-\infty$.

z is neither a real nor a complex number; it is the unit of a set of numbers that I call **zero-numbers**; these are numbers of the form:

$$cz$$

where c is a real number.

This allows for further extension of the set of complex numbers to the **hyper-complex numbers**.[2] Hyper-complex numbers are of the form:

$$h = a + bi + cz$$

where a, b, and c are real numbers. a is the real part $Re\{h\}$ of the hyper-complex number h; b is the imaginary part $Im\{h\}$, and c is the **zero-part** $Ze\{h\}$.

When $a = b = 0$ and $c \neq O$, h is a zero-number: $h = cz$.
When $b = c = 0$ and $a \neq O$, h is a real number: $h = a$.
When $a = c = 0$ and $b \neq O$, h is an imaginary number: $h = bi$.

Complex numbers can be represented in a plane (the complex plane), where the real numbers are on one axis (the x-axis) and the imaginary numbers are on the other axis (the y-axis). 1 is the unit of real numbers, i is the unit of imaginary numbers. Similarly, hyper-complex numbers can be represented in a three-dimensional space, as is shown in Figure 1. A z-axis is added to the complex plane for the zero-numbers.[3] z is the unit of the zero-numbers (the zero-unit) on the z-axis.

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[2] The term "hyper-complex numbers" is usually used for quaternions (four-dimensional extensions of complex numbers). I deviate from the usual nomenclature and refer to them as "meta-complex numbers" for four-dimensional numbers because they go beyond (*meta*) visual (three-dimensional) representation (see further).

[3] This happens to correspond with the symbol z for the zero unit.

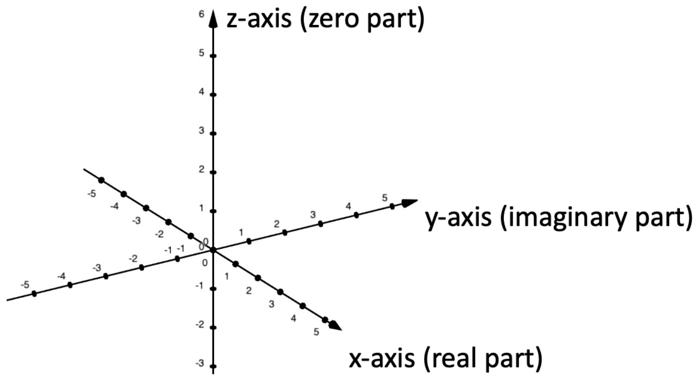

Fig. 1 visual representation of the axes of real, imaginary, and zero-numbers.

When $c = 0$ en $a \neq O$ en $b \neq O$, h is a complex number $h = a + bi$.
The set of complex numbers is ℂ.

When $b = 0$, $a \neq 0$ and $c \neq 0$, then h is a **real-zero number** $h = a + cz$.
The set of real-zero numbers is 𝔹.

When a = 0 and b ≠ 0 and c ≠ 0, then h is a **imaginary-zero number** h = bi + cz . The set of imaginary-zero numbers is 𝔸.

When a ≠ 0, b ≠ 0, and c ≠ 0, then h is a hyper-complex number h = a + bi + cz. The set of hyper-complex numbers is ℍ.

In the visual representation, the plane containing the x and y-axes represents the set of complex numbers ℂ, the plane determined by the x and z-axes represents the set of real-zero numbers 𝔹, and the plane determined by the y-axis and z-axis represents the set of imaginary-zero numbers 𝔸 (see Figure 2).

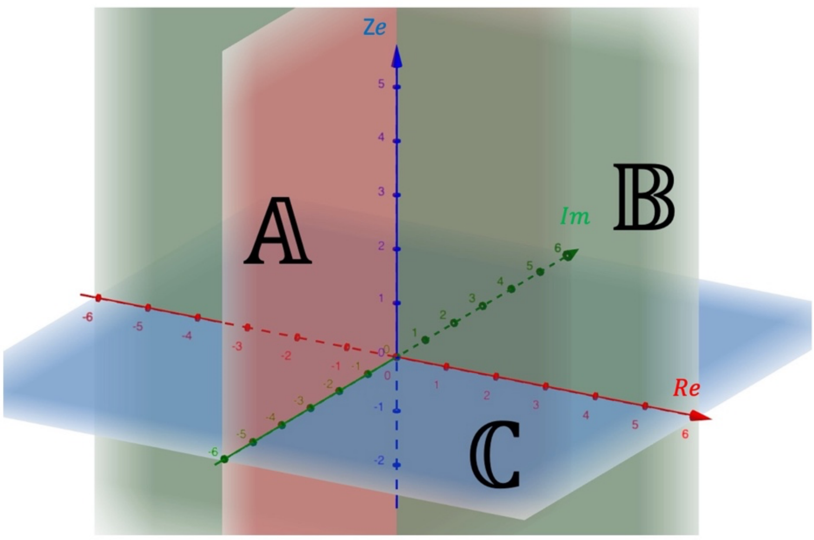

Fig. 2 Visual representation of the sets of imaginary-zero numbers (𝔸), real-zero numbers (𝔹), and complex numbers (ℂ) in the three-dimensional representation of the set of hyper-complex numbers(ℍ) .

Each hyper-complex number h can be represented by a point in the three-dimensional (hyper-complex) space. As an example, the number $h = 2 + 3i + 4z$ is shown in Figure 3.

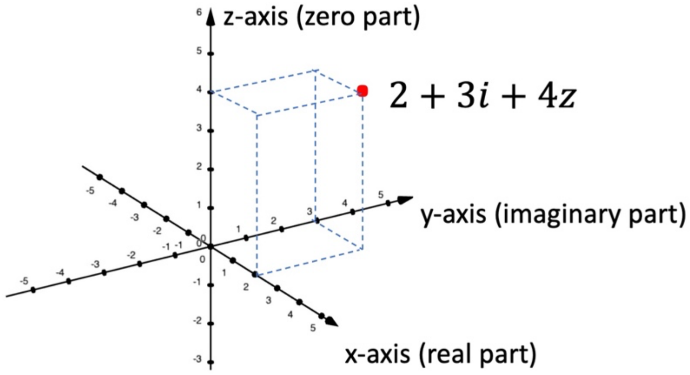

Fig. 3. Graphic representation of hyper-complex number $h = 2 + 3i + 4z$.

The following table provides an overview of the extended number sets.

a	real number	$\mathbb{R}$
bi	imaginary number	
cz	zero-number	
diz	meta-number	
$a + bi$	complex number	$\mathbb{C}$
$a + cz$	real-zero number	$\mathbb{B}$
$bi + cz$	imaginary-zero number	$\mathbb{A}$
$a + bi + cz$	hyper-complex number	$\mathbb{H}$
$a + bi + cz + \mathrm{diz}$	meta-complex number	$\mathbb{M}$

Table 1. Overview of number sets

3. Conjugates

The conjugate of a complex number $h = a + bi$ is the complex number $h^* = a - bi$. Similarly, we can define the conjugates of real-zero numbers, imaginary-zero numbers and hyper-complex numbers.

The conjugate of real-zero number $h = a + cz$ is $h^* = a - cz$

The conjugate of imaginary-zero number $h = bi + cz$ is $h^* = bi - cz$

The hyper-complex number $h = a + bi + cz$ has three possible conjugates:

$$h^{i*} = a - bi + cz$$

$$h^{z*} = a + bi - cz$$

and

$$h^{iz*} = a - bi - cz$$

4. Arithmetic with Hyper-complex Numbers; Meta-complex numbers

4.1 Addition and Subtraction

The sum (or difference if a, b, and/or c are negative real numbers) of two hyper-complex numbers h_1 and h_2 is obtained by adding the real part, imaginary part, and zero part of both numbers separately.

$$h_1 + h_2 = (a_1 + b_1 i + c_1 z) + (a_2 + b_2 i + c_2 z) = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)z$$

The sum of two hyper-complex numbers is again a hyper-complex number.

4.2 Multiplication

The product of two hyper-complex numbers h_1 and h_2 is obtained by multiplying each term of both numbers with each other.

$$h_1 . h_2 = (a_1 + b_1 i + c_1 z).(a_2 + b_2 i + c_2 z)$$

$$= a_1 a_2 + a_1 b_2 i + a_1 c_2 z + a_2 b_1 i + b_1 b_2 i^2 + b_1 c_2 iz + a_2 c_1 z + b_2 c_1 iz + c_1 c_2 z^2$$

$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i + (a_1 c_2 + a_2 c_1)z + (b_1 c_2 + b_2 c_1)iz + c_1 c_2 z^2$$

In this product, $a_1 a_2 - b_1 b_2$ is a real number (the real part), $(a_1 b_2 + a_2 b_1)i$ is an imaginary number, and $(a_1 c_2 + a_2 c_1)z$ is a zero-number. In addition to the real, imaginary, and zero parts, this product also contains a term (the fifth term) that includes z^2.

$$z^2 = z.z = \frac{1}{0}.\frac{1}{0} = \frac{1.1}{0.0} = \frac{1}{0}$$

This implies:

$$z^2 = z$$

The coefficient $(c_1 c_2)$ of the term with z^2 in the product can, therefore, be combined with the rest of the zero-part. The product $h_1 . h_2$ becomes:

$$h_1 . h_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i + (a_1 c_2 + a_2 c_1 + c_1 c_2)z + (b_1 c_2 + b_2 c_1)iz$$

Finally, there's the fourth term of the product in which i is multiplied by z:

$$iz = \sqrt{-1}\,.\frac{1}{0} = \frac{\sqrt{-1}}{0}$$

This number is neither an imaginary nor a zero number but belongs to the numbers we call **meta-numbers**. iz is the unit of meta-numbers. Meta-numbers are numbers of the form:

$$m = diz$$

Here, d is a real number.

In combination with the set of hyper-complex numbers, we obtain the set of **meta-complex numbers** $\mathbb{M}$. Meta-complex numbers are numbers of the form:

$$m = a + bi + cz + diz$$

where a, b, c, and d are real numbers. Meta-complex numbers cannot be visually represented in three-dimensional space, because they require a four-dimensional space.[4]

The product of two hyper-complex numbers is a meta-complex number.

4.3 Powers of z

We have already seen that $z^2 = z$. In general, for every $n \in \mathbb{N}_0$:

$$z^n = z$$

$$z^{-n} = \frac{1}{z^n} = \frac{1}{z} = \frac{0}{1} = 0$$

Just as the zeroth power of zero (0^0) leads to discussion because it is not uniquely defined[5], the zeroth power of the zero-unit (z^0) is not uniquely defined either. We can calculate z^0 in two ways:

$$z^0 = (\frac{1}{0})^0 = \frac{1^0}{0^0} = \frac{1}{1} = 1$$

or:

$$z^0 = z^{n-n} = \frac{z^n}{z^{-n}} = \frac{z}{0} = z.\frac{1}{0} = z.z = z^2 = z$$

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[4] Meta-complex numbers are comparable to quaternions (the extension of complex numbers to for dimensions) (see W. R. Hamilton).

[5] The most common values for $0^0$are 1 or "undefined" (because $0^0=\frac{0}{0}$, which results in division by zero).

Therefore, z^0 is undefined (or a choice has to be made, just like with real numbers).

4.3 Roots of z

$$\sqrt{z} = \sqrt{z^2} = z$$

In general, for every $n \in \mathbb{N}_0$:

$$\sqrt[n]{z} = z$$

4.4 Multiplication of conjugates

$$(a + cz).(a - cz) = aa - acz + acz - c^2z^2 = a^2 - c^2z$$

$$(bi + cz).(bi - cz) = b^2i^2 - bciz + bciz - c^2z^2 = -b^2 - c^2z$$

$$\begin{aligned}(a + bi + cz).(a - bi + cz) &= a^2 - abi + acz + abi - b^2i^2 + bciz + acz - bciz + c^2z^2 \\ &= a^2 + 2acz - b^2i^2 + c^2z^2 \\ &= a^2 + b^2 + (2ac + c^2)z\end{aligned}$$

$$\begin{aligned}(a + bi + cz).(a + bi - cz) &= a^2 + abi - acz + abi + b^2i^2 - bciz + acz + bciz - c^2z^2 \\ &= a^2 + 2abi + b^2i^2 + acz - c^2z^2 \\ &= a^2 - b^2 + 2abi + (ac - c^2)z\end{aligned}$$

$$\begin{aligned}(a + bi + cz).(a - bi - cz) &= a^2 - abi - acz + abi - b^2i^2 - bciz + acz - bciz - c^2z^2 \\ &= a^2 - b^2i^2 - 2bciz - c^2z^2 \\ &= a^2 + b^2 - c^2z - 2bciz\end{aligned}$$

4.4 Division

A hyper-complex or meta-complex number can be divided by a real number or a complex number, but the division by a hyper-complex or meta-complex number is undefined in $\mathbb{H}$ and $\mathbb{M}$. The quotient of a meta-complex number divided by a real number r is:

$$\frac{a + bi + cz + diz}{r} = \frac{a}{r} + \frac{b}{r}i + \frac{c}{r}z + \frac{d}{r}iz$$

This is again a meta-complex number.

A meta-complex number can also be divided by zero:

$$\frac{a + bi + cz + diz}{0} = \frac{a}{0} + \frac{b}{0}i + \frac{c}{0}z + \frac{d}{0}iz = az + biz + cz^2 + diz^2 = (a + c)z + (b + d)iz$$

The division of a meta-complex number by a complex number is:

$$\frac{a+bi+cz+diz}{p+qi}=\frac{(a+bi+cz+diz).(p-qi)}{(p+qi).(p-qi)}$$

$$=\frac{ap-aqi+bpi-bqi^2+cpz-cqiz+dpiz-dqi^2z)}{p^2-q^2i^2}$$

$$=\frac{(ap+bq)+(-aq+bp)i+(cp+dq)z+(-cq+dp)iz}{p^2+q^2}$$

$$=\frac{ap+bq}{p^2+q^2}+(\frac{ap+bq}{p^2+q^2})i+(\frac{ap+bq}{p^2+q^2})z+(\frac{ap+bq}{p^2+q^2})iz$$

This is again a meta-complex number.

5. The Ring of meta-numbers

The set of meta-complex numbers with addition and multiplication has the following characteristics:

5.1 Closure under addition and multiplication

For all m_1 and $m_2 \in \mathbb{M}$: $m_1 + m_2 \in \mathbb{M}$ (see 4.1).

For all m_1 and $m_2 \in \mathbb{M}$: $m_1 . m_2 \mathbb{M}$ (see 4.2).

Note that there is no closure under multiplication for the set $\mathbb{H}$ of hyper-complex numbers, because the product of two hyper-complex numbers is not generally a hyper-complex number, but a meta-complex number.

5.2 Commutativity of addition and multiplication

a) Addition is commutative in $\mathbb{M}$

For all m_1 and $m_2 \in \mathbb{M}$: $m_1 + m_2 = m_2 + m_1$

proof:

$m_1 + m_2 = (a_1 + b_1 i + c_1 z + d_1 iz) + (a_2 + b_2 i + c_2 z + d_2 iz)$

$= (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)z + (d_1 + d_2)iz$

and

$m_2 + m_1 = (a_2 + b_2 i + c_2 z + d_2 iz) + (a_1 + b_1 i + c_1 z + d_1 iz)$

$= (a_2 + a_1) \ + (b_2 + b_1)i + (c_2 + c_1)z + (d_2 + d_1)iz$

$= (a_1 + a_2) \ + (b_1 + b_2)i + (c_1 + c_2)z + (d_1 + d_2)iz$

$\Rightarrow m_1 + m_2 = \ m_2 + m_1$ ■

b) Multiplication is commutative in $\mathbb{M}$.

For all m_1 and $m_2 \in \ \mathbb{M}$: $m_1.m_2 = \ m_2.m_1$

proof:

$m_1.m_2 = (a_1 + b_1i + c_1z + d_1iz).(a_2 + b_2i + c_2z + d_2iz)$

$$= (a_1a_2 + a_1b_2i + a_1c_2z + a_1d_2iz) + (a_2b_1i + b_1b_2i^2 + b_1c_2iz + b_1d_2i^2z) + (a_2c_1z + b_2c_1iz + c_1c_2z^2 + c_1d_2iz^2) + (a_2d_1iz + b_2d_1i^2z + c_2d_1iz^2 + d_1d_2i^2z^2)$$

$$= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i + (a_1c_2 + a_2c_1 - b_1d_2 - b_2d_1 + c_1c_2 - d_1d_2)z + (a_1d_2 + a_2d_1 + b_1c_2 + b_2c_1 + c_1d_2 + c_2d_1)iz$$

and

$m_2 + m_1 = (a_2 + b_2i + c_2z + d_2iz) + (a_1 + b_1i + c_1z + d_1iz)$

$$= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i + (a_1c_2 + a_2c_1 - b_1d_2 - b_2d_1 + c_1c_2 - d_1d_2)z + (a_1d_2 + a_2d_1 + b_1c_2 + b_2c_1 + c_1d_2 + c_2d_1)iz$$

$\Rightarrow m_1.m_2 = \ m_2.m_1$ ■

5.3 Additive and multiplicative identity element

a) Additive identity element

$0 + 0i + 0z + 0iz$ (or 0 for short) is the additive identity element of $\mathbb{M}$.

For all $m \in \mathbb{M}$: $m + (0 + 0i + 0z + 0iz) \ = m + 0 = m$

Proof:

$\forall a, b, c \in \mathbb{R}$:

$m + (0 + 0i + 0z + 0iz) \ = (a + bi + cz + diz) + (0 + 0i + 0z + 0iz)$

$= (a + bi + cz + diz) + 0 = a + bi + cz + diz = m$ ■

b) Multiplicative identity element

The real number 1 is the multiplicative identity element of $\mathbb{M}$.

For all $m \in \mathbb{M}$: $1.m = m$

Proof:

$\forall a, b, c \in \mathbb{R}$:

$$1.m = 1.(a + bi + cz + diz) = a + bi + cz + diz = m \;\blacksquare$$

5.4 Additive inverses

$-a - bi - cz - diz$ is the inverse element of any meta-complex number.

For all $m \in \mathbb{M}$: $m + (-m) = m - m = 0$

Proof
$\forall\, m \in \mathbb{M}$:

$m + -m = (a + bi + cz + diz) + (-a - bi - cz - diz)$

$= (a + bi + cz + diz) - (a + bi + cz + diz) = 0 \;\blacksquare$

5.6 Associativity of addition and multiplication

For all m_1, m_2 and $m_3 \in \mathbb{M}$:

$$(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3) = m_1 + m_2 + m_3$$

proof:
$\forall\, m_1, m_2$ and $m_3 \in \mathbb{M}$:

$(m_1 + m_2) + m_3$

$= [(a_1 + b_1 i + c_1 z + d_1 iz) + (a_2 + b_2 i + c_2 z + d_2 iz)] + (a_3 + b_3 i + c_3 z + d_3 iz)$

$= [(a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)z + (d_1 + d_2)iz] + (a_3 + b_3 i + c_3 z + d_3 iz)$

$= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i + (c_1 + c_2 + c_3)z + (d_1 + d_2 + d_3)iz$

and similarly:

$m_1 + (m_2 + m_3)$

$= (a_1 + b_1 i + c_1 z + d_1 iz) + [(a_2 + b_2 i + c_2 z + d_2 iz) + (a_3 + b_3 i + c_3 z + d_3 iz)]$

$= (a_1 + a_2 + a_3) \ + (b_1 + b_2 + b_3)i + (c_1 + c_2 + c_3)z + (d_1 + d_2 + d_3)iz$

and similarly:

$m_1 + m_2 + m_3$

$= (a_1 + b_1 i + c_1 z + d_1 iz) + (a_2 + b_2 i + c_2 z + d_2 iz) + (a_3 + b_3 i + c_3 z + d_3 iz)$

$= (a_1 + a_2 + a_3) \ + (b_1 + b_2 + b_3)i + (c_1 + c_2 + c_3)z + (d_1 + d_2 + d_3)iz$

$\Rightarrow (m_1 + m_2) + m_3 = \ m_1 + (m_2 + m_3) = m_1 + m_2 + m_3$ ■

5.7 Distributivity of multiplication and addition

For all m_1, m_2, and $m_3 \in \mathbb{M}$: $m_1.(m_2 + m_3) = m_1.m_2 + m_1.m_3$

proof:
$\forall\ m_1, m_2$ and $m_3 \in \mathbb{M}$:

$m_1.(m_2 + m_3)$

$= (a_1 + b_1 i + c_1 z + d_1 iz).[(a_2 + b_2 i + c_2 z + d_2 iz) + (a_3 + b_3 i + c_3 z + d_3 iz)]$

$= (a_1 + b_1 i + c_1 z + d_1 iz).[(a_2 + a_3) + (b_2 + b_3)i + (c_2 + c_3)z + (d_2 + d_3)iz]$

$= a_1.(a_2 + a_3) + a_1.(b_2 + b_3)i + a_1.(c_2 + c_3)z + a_1.(d_2 + d_3)iz$
$+ b_1 i.(a_2 + a_3) + b_1 i.(b_2 + b_3)i + b_1 i.(c_2 + c_3)z + + b_1 i.(d_2 + d_3)iz$
$+ c_1 z.(a_2 + a_3) + c_1 z.(b_2 + b_3)i + c_1 z.(c_2 + c_3)z + c_1 z.(d_2 + d_3)iz$

$= a_1 a_2 + a_1 a_3 + (a_1 b_2 + a_1 b_3 + a_2 b_1 + a_3 b_1)i + (a_1 c_2 + a_1 c_3 + a_2 c_1 + a_3 c_1)z$
$+ (b_1 b_2 + b_1 b_3)i^2 + (b_1 c_2 + b_1 c_3 + b_2 c_1 + b_3 c_1)iz + (c_1 c_2 + c_1 c_3)z^2 + (a_1 d_2 + a_1 d_3)iz$
$+ (b_1 d_2 + b_1 d_3)i^2 z + (c_1 d_2 + c_1 d_3)iz^2$

$= (a_1 a_2 + a_1 a_3 - b_1 b_2 - b_1 b_3) + (a_1 b_2 + a_1 b_3 + a_2 b_1 + a_3 b_1)i + (a_1 c_2 + a_1 c_3 + a_2 c_1$
$+ a_3 c_1 - b_1 d_2 - b_1 d_3 + c_1 c_2 + c_1 c_3)z + (a_1 d_2 + a_1 d_3 + b_1 c_2 + b_1 c_3 + b_2 c_1 + b_3 c_1$
$+ c_1 d_2 + c_1 d_3)iz$

and

$m_1.m_2 + m_1.m_3$

$= [(a_1 + b_1 i + c_1 z + d_1 iz).(a_2 + b_2 i + c_2 z + d_2 iz)]$
$+ [(a_1 + b_1 i + c_1 z + d_1 iz).(a_3 + b_3 i + c_3 z + d_3 iz)]$

$= (a_1 a_2 + a_1 b_2 i + a_1 c_2 z + a_1 d_2 iz) + (b_1 a_2 i + b_1 b_2 i^2 + b_1 c_2 iz + b_1 d_2 i^2 z)$

$$+(a_2c_1z + b_2c_1iz + c_1c_2z^2 + c_1d_2iz^2) + \ (a_2d_1iz + b_2d_1i^2z + c_2d_1iz^2 + d_1d_2i^2z^2)$$
$$+(a_1a_3 + a_1b_3i + a_1c_3z + a_1d_3iz) + (b_1a_3i + b_1b_3i^2 + b_1c_3iz + b_1d_3i^2z)$$
$$+(a_3c_1z + b_3c_1iz + c_1c_3z^2 + c_1d_3iz^2) + \ (a_3d_1iz + b_3d_1i^2z + c_3d_1iz^2 + d_1d_3i^2z^2)$$

$$= (a_1a_2 + a_1b_2i + a_1c_2z + a_1d_2iz) + (b_1a_2i - b_1b_2 + b_1c_2iz - b_1d_2z)$$
$$+(a_2c_1z + b_2c_1iz + c_1c_2z + c_1d_2iz) + \ (a_2d_1iz - b_2d_1z + c_2d_1iz - d_1d_2z)$$
$$+(a_1a_3 + a_1b_3i + a_1c_3z + a_1d_3iz) + (b_1a_3i - b_1b_3 + b_1c_3iz - b_1d_3z)$$
$$+(a_3c_1z + b_3c_1iz + c_1c_3z + c_1d_3iz) + \ (a_3d_1iz - b_3d_1z + c_3d_1iz - d_1d_3z)$$

$$= (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3) + (a_1b_2 + a_1b_3 + a_2b_1 + a_3b_1)i + (a_1c_2 + a_1c_3 + a_2c_1$$
$$+a_3c_1 - b_1d_2 - b_1d_3 + c_1c_2 + c_1c_3)z + (a_1d_2 + a_1d_3 + b_1c_2 + b_1c_3 + b_2c_1 + b_3c_1$$
$$+c_1d_2 + c_1d_3)iz$$

$$\Rightarrow m_1.(m_2 + m_3) = m_1.m_2 + m_1.m_3 \ \blacksquare$$

Therefore, the set of meta-complex numbers with addition and multiplication is a **commutative ring with unity.**

6. The norm of meta-complex numbers

The norm of a meta-complex number $m = a + bi + cz + diz$ is defined as:

$$\parallel m \parallel = \sqrt{a^2 + b^2 + c^2 + d^2}$$

This is the length of the vector running from the origin (0,0,0,0) to m in the four-dimensional space of meta-complex numbers.

7. homotope zeros

0 is a real number. $0i$ is an imaginary number. $0z$ is a zero-number. $0iz$ is a meta-number. Even though the four zeros occupy the same position in the four-dimensional space (the origin), they are distinct. I refer to them as **homotopic numbers**.

$$0 \neq 0i \neq 0z \neq 0iz$$

The (real) number 0 is reached by moving along the x-axis to the origin of the coordinate system (both from the left and from the right). for meta-complex number $a + bi + cz + diz$:

$$0 = \lim_{a \to 0} a$$

$$0i = \lim_{b \to 0} b$$

$$0z = \lim_{c \to 0} c$$

$$0iz = \lim_{d \to 0} c$$

It also follows that $\frac{0}{0}$ is not a real number but a zero-number. Indeed:

$$\frac{0}{0} = 0z \; (\neq 0\,!)$$

When the coefficients b, c, and/or d in $a + bi + cz + diz$ are equal to zero, they can be omitted in the notation. So, although $0 \neq 0i \neq 0z \neq 0iz$, in most cases the meta-complex number $0 + 0i + 0z + 0iz$ can be written as 0 without causing confusion.

8. Hyperc-omplex numbers in spherical coordinates

The coefficients of the hyper-complex number $h = a + bi + cz$ in spherical coordinates are as follows:

$$a = r.sin\theta.cos\varphi$$
$$b = r.sin\theta.sin\varphi$$
$$c = r.cos\varphi$$

Here, φ is the angle between the real axis (the x-axis) and the projection of $\vec{r}$ onto the complex plane (set $\mathbb{C}$), measured from the x-axis; θ is the angle between the zero axis (z-axis) and $\vec{r}$ measured from the z-axis (see Figure 4).

We can write h as:

$$h = r(sin\theta.cos\varphi + i.sin\theta.sin\varphi + z.cos\varphi)$$

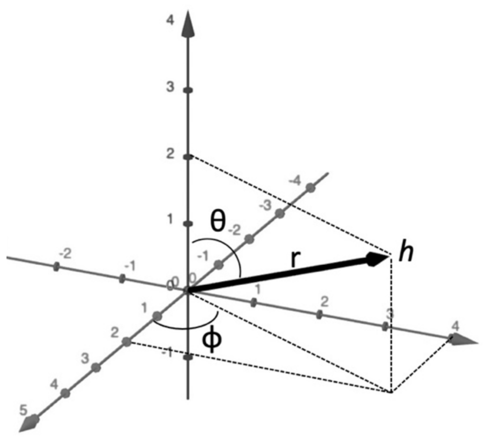

Fig. 4. hyper-complex number h in spherical coordinates

This results in the following extension of Euler's formula:

$$h = r(e^{i\varphi}.sin\theta + z.cos\theta)$$

proof:

$h = r(sin\theta.cos\varphi + i.sin\theta.sin\varphi + z.cos\theta) = r(sin\theta(cos\varphi + i.sin\varphi) + z.cos\theta$

and since: $cos\varphi + i.sin\varphi = e^{i\varphi}$ (Euler's Formula):

$h = r(e^{i\varphi}.sin\theta + z.cos\theta)$ ■

9. Some applications of hyper-complex numbers

As a result of introducing zero-numbers, rational expressions (a quotient of two polynomials) as well as functions where division by zero occurs have solutions for all values of ℝ. Here are some examples to illustrate this.

$f(x) = \frac{1}{x} \quad \Rightarrow \quad f(0) = \frac{1}{0} = z$

$f(x) = \frac{5}{x} \quad \Rightarrow \quad f(0) = \frac{5}{0} = 5z$

$f(x) = \frac{2x+3}{x-1} \quad \Rightarrow \quad f(1) = \frac{2+3}{0} = 5z$

$f(x) = \frac{5x-1}{x-1} \quad \Rightarrow \quad f(1) = \frac{5-1}{0} = 4z$

$f(x) = \frac{1}{sinx} \quad \Rightarrow \quad f(k.\pi) = \frac{1}{sin(k.\pi)} = z$ (for k ∈ ℤ)

For $\alpha \in \mathbb{R}$: $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ is undefined in ℝ when $\cos\alpha = 0$. This is the case when $\alpha = \frac{k\pi}{2}$ (for k ∈ ℤ). In ℍ however:

$$\tan\frac{k\pi}{2} = \sin\frac{k\pi}{2}.\frac{1}{\cos\frac{k\pi}{2}} = 1.\frac{1}{0} = 1.z = z$$

Similarly:

$$\cot 0 = z$$

$$\tan\frac{3\pi}{2} = -z$$

$$\cot\pi = -z$$

10. Folding back of functions

We already saw that for positive values of x approaching 0, $f(x) = \frac{1}{x}$ gets closer to ∞, while for negative values of x approaching 0, $\frac{1}{x}$ gets closer to $-\infty$ in $\mathbb{R}$. So the function $f(x) = \frac{1}{x}$ has no real solution for $x = 0$. In $\mathbb{H}$ however, $f(0) = \frac{1}{0} = z$ (see Figure 5).

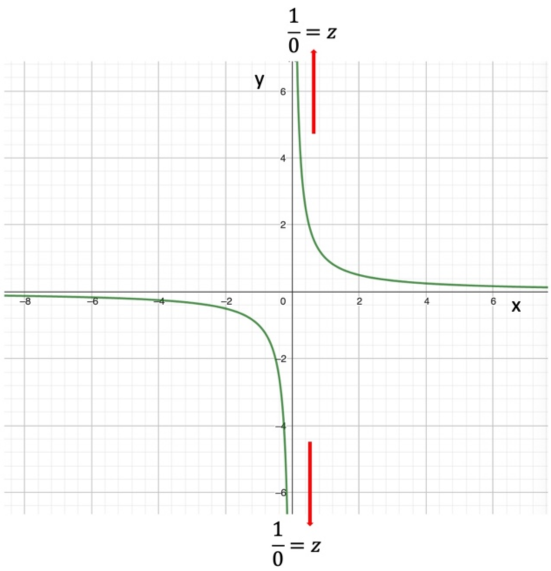

Fig. 5. $f(x) = \frac{1}{x}$ for x approaching 0.

We could interpret this as if the function $f(x) = \frac{1}{x}$ **"folds back"** into z for $x = 0$ (see Figure 6).

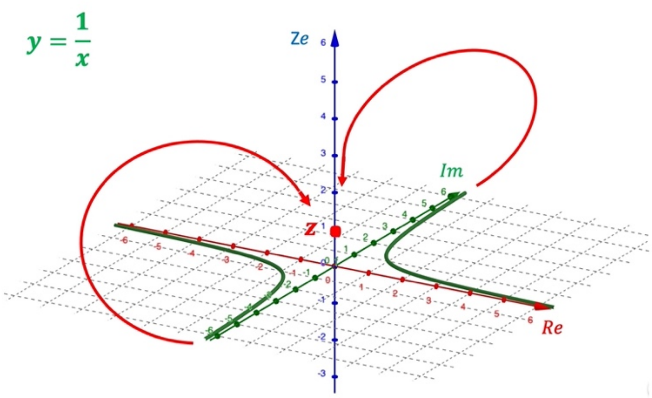

Fig. 6. $f(x) = \frac{1}{x}$ folds back into Z for $x = 0$.

Similarly, $f(x) = \frac{5}{x}$ folds back into $5z$ for $x = 0$ (see Figure 7).

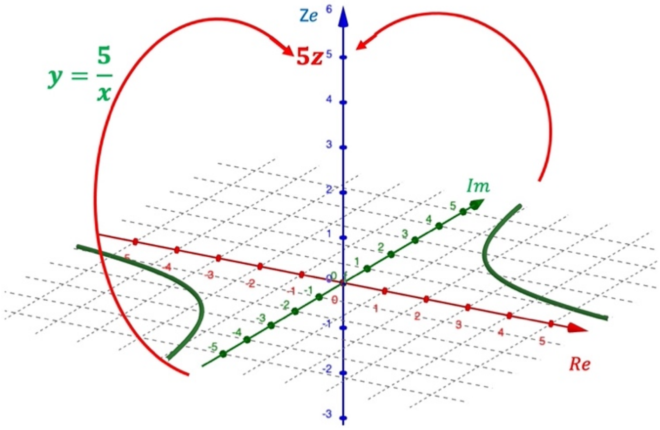

Fig. 7. $f(x) = \frac{1}{x}$ folds back into z for $x = 0$.

Whenever a function results in division by zero for some value, the function folds back into a zero-number dz (where d is a real number).

11. Collapsing and spreading of number sets

When for any hyper-complex number $h = a + bi + cz$, coefficient c approaches zero, at the limit, the number becomes a complex number.

$$\lim_{c \to 0} a + bi + cz = a + bi$$

We could interpret this as the set $\mathbb{Z}$ of zero-numbers "**collapsing**" and "**spreading**" into the set $\mathbb{C}$ of complex numbers (see Figure 8).

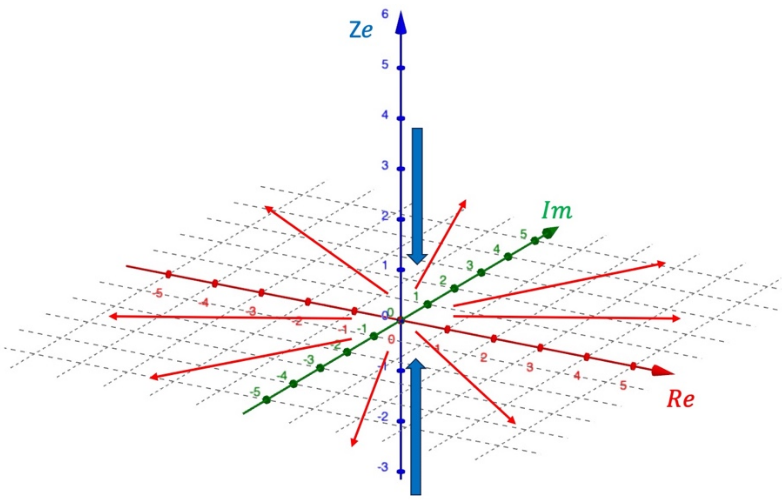

Fig. 8 Set $\mathbb{Z}$ of zero-numbers "collapsing" (blue arrows) and "spreading" (red arrows) into the set $\mathbb{C}$ of complex numbers.

Similarly, the set ℝ of real numbers collapses and spreads into the set 𝔸 of imaginary-zero numbers, and the set of imaginary numbers collapses and spreads into the set 𝔹 of real-zero numbers. Any set of numbers may collapse and spread into another set that does not include the original set, as if through a "wormhole" to another dimension.

11. Conclusion and further developments

The theory developed above is far from complete. The study is merely intended to introduce number sets that allow for division by zero. This way, it aims to expand the possibilities of arithmetic. It is my belief that the theory can be further developed. Whether hyper-complex and meta-complex numbers also have practical applications remains to be seen. In any case, it has the potential to enrich the array of mathematical tools.

It doesn't seem impossible that the hyper-complex and meta-complex numbers could find an application, for instance, in quantum mechanics, where the use of complex numbers is already a necessity. The hyper-complex and meta-complex numbers might serve as a tool for studying phenomena and conditions that approach zero or infinity (such as singularities, black holes, vacuum energy, etc.). The future (in space-time) will tell...

08 September 2023